NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2023 Year 12 Course Assessment Task 4 (Trial Examination)
Wednesday, 16 August 2023

General instructions

- Working time – 3 hours. (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt all questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I - 10 marks

- Mark your answers on the answer grid provided.
- Attempt Questions 1–10
- Allow about 15 minutes for this section

SECTION II - 90 marks

- Commence each new question on a new booklet. Write on both sides of the paper.
- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED:** ....

Class: (please ✓)

○ 12MXX.1 – Mr Ho
○ 12MXX.2 – Mr Sekaran
○ 12MXX.3 – Ms Ham

Marker's use only

QUESTION	1-10	11	12	13	14	15	16	Total	%
MARKS	$\frac{}{10}$	$\frac{}{13}$	$\frac{}{13}$	$\frac{}{12}$	$\frac{}{17}$	$\frac{}{18}$	$\frac{}{17}$	$\frac{}{100}$	

Section I

10 marks
Attempt Questions 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer grid provided.

Questions **Marks**

1. Which of the following expression best represents $4 + 3i$? **1**

(A) $5e^{36.679\,i}$ (B) $25e^{0.644\,i}$ (C) $5e^{0.927\,i}$ (D) $5e^{0.644\,i}$

2. The complex numbers $z,\ iz$ and $z + iz$, where z is a non-zero complex number, are plotted in the Argand plane, forming the vertices of a triangle. **1**

Which of the following is the area of the triangle?

(A) $|z|$

(B) $|z| + |z|^2$

(C) $\frac{|z|^2}{2}$

(D) $\frac{\sqrt{3}}{2}|z|^2$

3. Which of the following statement is true? **1**

(A) $\forall\ x \in \mathbb{R}, \exists\ y \in \mathbb{R}$ such that $xy = 10$

(B) $\exists\ x \in \mathbb{R}$, such that $\forall\ y \in \mathbb{R}, x + y = 10$

(C) $\exists\ x \in \mathbb{R}$, such that $\forall\ y \in \mathbb{R}, xy = 10$

(D) $\forall\ x \in \mathbb{R}, \exists\ y \in \mathbb{R}$ such that $x + y = 10$

4. Which of the following is the Cartesian equation for a sphere with centre $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and radius 3? **1**

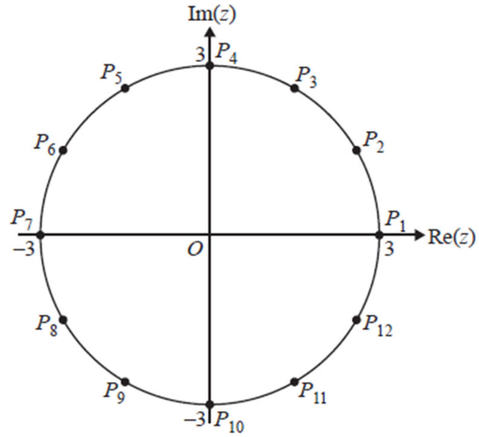
(A) $x^2 + x + y^2 + 2y + z^2 - 2z = 0$

(B) $x^2 - x + y^2 - 2y + z^2 + 2z = 0$

(C) $x^2 + 2x + y^2 + 4y + z^2 - 4z = 0$

(D) $x^2 - 2x + y^2 - 4y + z^2 + 4z = 0$

5. On the argand diagram below, the twelve points $P_1, P_2, P_3, \cdots, P_{12}$ are evenly spaced around the circle of radius 3. **1**



Which set of points represent the solutions to $z^3 = -27\,i$?

(A) P_2, P_6, P_{11} (B) P_4, P_8, P_{12} (C) P_3, P_7, P_{11} (D) P_1, P_5, P_9

6. What is the value of $\displaystyle\int_{-k}^{k} \{f(x) - f(-x)\}\, dx$? **1**

(A) $\displaystyle\int_0^k f(x)\, dx$

(B) $4\displaystyle\int_0^k f(x)\, dx$

(C) 0

(D) $2\displaystyle\int_0^k f(x)\, dx$

7. Which expression is equal to $\displaystyle\int x\sqrt{1-x}\, dx$? **1**

(A) $-\frac{1}{3}x^2(1-x)^{\frac{3}{2}} + c$

(B) $\frac{1}{3}x^2(1-x)^{\frac{3}{2}} + c$

(C) $-\frac{2}{5}x(1-x)^{\frac{5}{2}} + \frac{2}{3}x(1-x)^{\frac{3}{2}} + c$

(D) $\frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + c$

Examination continues overleaf. . .

8. Suppose $z = p + iq$ is a solution of the polynomial equation **1**

$$c_4 z^4 + i c_3 z^3 + c_2 z^2 + i c_1 z + c_0 = 0$$

where p, q, c_4, c_3, c_2, c_1 and c_0 are real.

Which of the following must also be a solution?

(A) $q + ip$ (B) $-p + iq$ (C) $-p - iq$ (D) $-p - iq$

9. The line ℓ_1 has vector equation $\underset{\sim}{r_1} = \underset{\sim}{i} + \lambda(\underset{\sim}{j} - \underset{\sim}{k})$ and the line ℓ_2 has vector equation **1**

$\underset{\sim}{r_2} = \left(3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right) + \mu\left(2\underset{\sim}{i} + 2\underset{\sim}{k}\right)$, where $\lambda, \mu \in \mathbb{R}$.

Which of the following statements is correct?

(A) ℓ_1 and ℓ_2 are parallel.

(B) ℓ_1 and ℓ_2 intersect at a point.

(C) ℓ_1 and ℓ_2 are perpendicular.

(D) ℓ_1 and ℓ_2 are skew.

10. A ball is thrown vertically up with an initial velocity of $7\sqrt{6}\ \text{ms}^{-1}$, and it is subject to gravity and air resistance. The acceleration of the ball is given by $\ddot{x} = -(9.8 + 0.1\,v^2)\text{ms}^{-2}$, where x metres is its vertical displacement from the point of projection, and $v\ \text{ms}^{-1}$ is its velocity at time t seconds. **1**

Which of the following is the time, in seconds, taken for the ball to reach its maximum height?

(A) $\frac{10\pi}{21\sqrt{2}}$

(B) $\frac{5\pi}{21\sqrt{2}}$

(C) $\log_e 4$

(D) $\frac{\pi}{3}$

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (13 marks) **Commence a New booklet** **Marks**

(a) If $a + bi = \dfrac{2+4i}{1-i}(b+i)$, where a and b are real constants, find the values of a and b. **2**

(b) For real numbers $a, b > 0$ prove that $\frac{a}{b} + \frac{b}{a} \geq 2$. **2**

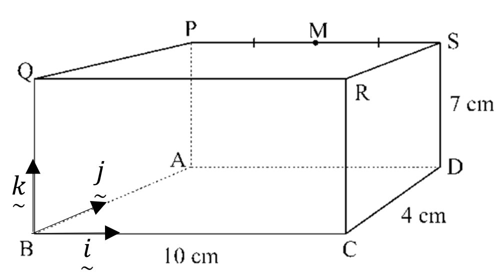
(c) The complex number z is given by $z = -p + pi$, where p is a positive real number.

It is given that $w = \dfrac{\sqrt{2}\,\bar{z}}{z^4}$.

i. Express w in the form $re^{i\theta}$, in terms of p, where $r > 0$ and $-\pi < \theta \leq \pi$. **2**

ii. Find the smallest positive whole number n such that $\text{Re}\,(w^n) = 0$. **2**

(d) The unit vectors along $\overrightarrow{BC}, \overrightarrow{BA}$, and $\overrightarrow{BQ}$ are $\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k}$ respectively with each of its magnitude 1 cm. M is the midpoint of PS of the rectangular prism.



i. Find $\overrightarrow{DM}$. **1**

ii. Using vector method, find the size of $\angle QDM$, giving your answer to the nearest degree. **2**

(e) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,\cos^3 x\; dx$ **2**

Examination continues overleaf. . .

Question 12 (13 marks) **Commence a New booklet** **Marks**

(a) Consider the statement:

$$\forall x \in \mathbb{R},\ (x \geq 3) \Rightarrow (x^2 > 5)$$

i. Write the contrapositive. **1**

ii. Write the negation. **1**

(b) A motorist is travelling at a constant speed of $20\ \text{ms}^{-1}$. When the motorist reaches a horizontal section of the road, the brakes are applied. The combined retarding force from the brakes and the friction of the road, is proportional to the speed v of the car. After travelling 80 metres along this section of the road, the speed of the car has fallen to 10 ms^{-1}.

Let x metres be the distance of the car from the start of the horizontal section.

i. Show that $\ddot{x} = -kv$, where k is a constant. **1**

ii. Find the value of k. **2**

iii. How long did it take for the speed to drop from 20 ms^{-1} to 10 ms^{-1}? Give the answer as an exact value. **2**

(c) Using partial fractions, show that $\displaystyle\int_0^1 \frac{5(1-x)}{(1+x)(3-2x)}\,dx = \ln\frac{4}{\sqrt{3}}$. **4**

(d) i. Sketch the graph of the set of points z defined by $|z-(3-4\,i)| = 3$, where $z \in \mathbb{C}$. **1**

ii. P is a point on the graph drawn in part (i) such that the modulus of the complex number represented by P is the smallest. **1**

Find the complex number represented by P in $a+ib$ form.

Question 13 (12 marks) **Commence a New booklet** **Marks**

(a) For integers a and b, prove that if $a+b$ is odd then a^2+b^2 is odd. **2**

(b) Let $f(x)=x-\ln(1+x)$ and $g(x)=x+\ln(1-x)$, where $0 \le x < 1$.

i. By differentiating $f(x)$, show that $\ln(1+x) < x$ for $0 < x < 1$. **2**

ii. By differentiating $g(x)$, show that $-\ln(1-x) > x$ for $0 < x < 1$. **2**

iii. Deduce from (i) and (ii) that **2**

$$\ln(n+1) - \ln n < \frac{1}{n} < \ln n - \ln(n-1)$$

for all positive integer $n > 1$.

iv. Hence or otherwise show that $6.21 < \sum_{k=2}^{1000} \frac{1}{k} < 6.91$. **2**

(c) One of the roots of the equation $3z^3+13z^2+20z+14=0$ is $-1+i$. **2**

Find the other roots of the equation.

Examination continues overleaf. . .

Question 14 (17 marks) **Commence a New booklet** **Marks**

(a) i. The line ℓ_1 has Cartesian equation $x = -y = \frac{z}{2}$. **1**

Show that its vector equation is $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, where $\lambda \in \mathbb{R}$.

ii. Write the vector equation of the line ℓ_2 in the form $\underset{\sim}{r} = \underset{\sim}{a} + \mu \underset{\sim}{b}$ that passes through the point $A\ (1, 1, 0)$ and is parallel to the vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, where $\mu \in \mathbb{R}$. **1**

iii. Find the acute angle θ between the lines ℓ_1 and ℓ_2 **1**

iv. Find the coordinates of the point of intersection N of the lines ℓ_1 and ℓ_2 **1**

v. Find the shortest distance from the point A to the line ℓ_1. **2**

vi. Find the equation of a line ℓ_3 which bisects the acute angle θ and passes through N such that the three lines ℓ_1, ℓ_2 and ℓ_3 lie on the same plane. **2**
[You may consider the unit vectors of the directional vectors of lines ℓ_1 and ℓ_2].

(b) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$, for $n \in \mathbb{Z}^+$.

i. Using integration by parts, show that $nI_n = (n-1)I_{n-2}$, where $n \geq 2$. **2**

A sequence $\{x_n\}$ is defined by $x_n = nI_nI_{n-1}$ for $n \in \mathbb{Z}^+$.

ii. Using part (i) or otherwise, show that $x_{n+1} = x_n$. **1**

iii. Hence show that $x_1 = x_2 = x_3 = \cdots\cdots = x_n = x_{n+1} = \frac{\pi}{2}$. **2**

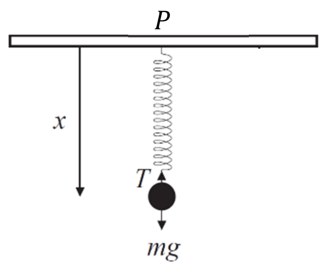
iv. Explain why $I_n \leq I_{n-1}$. **1**

v. Using parts (ii), (iii) and (iv) or otherwise, show $\sqrt{\frac{\pi}{2(n+1)}} \leq I_n \leq \sqrt{\frac{\pi}{2n}}$ for $n \in \mathbb{Z}^+$. **3**

Question 15 (18 marks) **Commence a New booklet** **Marks**

(a) One end of a light spring of natural length l m is tied to a fixed point P at the ceiling and the other end to a particle of mass m kg as shown in the diagram below.

Assume there is no air resistance. g is the constant acceleration due to gravity.



The particle is initially pulled down to a distance of 6 m from the ceiling and from there it is projected downwards at a speed of 3.5 ms^{-1}. The particle then oscillates vertically in simple harmonic motion.

Let x be the displacement of the particle from the ceiling.

For the particle, the force T exerted by the spring is proportional to $x - l$, that is $T = k(x - l)$ newtons where k is the stiffness of the spring which is a positive constant.

i. Show that $\ddot{x} = -\frac{k}{m}\left[x - \left(\frac{mg}{k} + l\right)\right]$. **1**

ii Show that $x = a\cos\left(\sqrt{\frac{k}{m}}\,t + \alpha\right) + \frac{mg}{k} + l$ satisfies the differential equation in part (i), where a and α are constants. **1**

It is given that $m = 4$ kg, $l = 5$ m, $k = 49$ Nm^{-1} and $g = 9.8$ ms^{-2}.

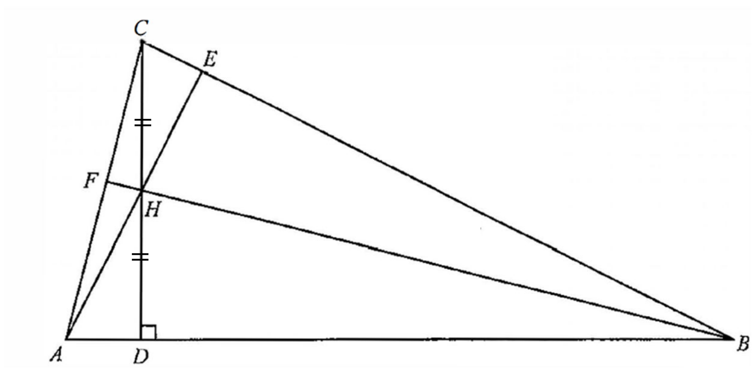
iii. Find the center of motion. **1**

iv Find the period and amplitude of the motion. **3**

v. Find the values of T at the extremes of the motion. **2**

Examination continues overleaf. . .

(b) **Marks**



CD is an altitude of the ΔABC and H is a mid point of CD. AH and BH are produced to meet BC and AC at E and F respectively.

Let $\underset{\sim}{p}$, $\lambda \underset{\sim}{p}$ $(\lambda > 1)$ and $\underset{\sim}{q}$ be $\overrightarrow{AD}$, $\overrightarrow{AB}$ and $\overrightarrow{DH}$ respectively. Let $\frac{BE}{EC} = r$.

i. Find $\overrightarrow{AC}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$. **1**

ii. Show that $\overrightarrow{AE} = \dfrac{(r+\lambda)\underset{\sim}{p} + 2r\underset{\sim}{q}}{1+r}$. **2**

iii. Using the fact that A, H and E are collinear, show that $r = \lambda$. **2**

It is given that $|\underset{\sim}{p}| = 1$ and $|\underset{\sim}{q}| = 2$ and H is the orthocentre of the triangle ΔABC.

[The orthocenter is the point where all the three altitudes of the triangle intersect each other].

iv. Using AH is perpendicular to BC or otherwise, find the value of λ. **2**

v. Using the fact that B, H and F are collinear or otherwise, find the ratio $AF : FC$ **3**

Question 16 (17 marks) **Commence a New booklet** **Marks**

(a) A food parcel is dropped vertically from a rescue helicopter which is 2000 metres above a group of stranded refugees in a war-torn country. After 10 seconds a parachute opens automatically.
Air resistance is neglected for the first 10 seconds but then the effect of the open parachute applies a resistance of $2Mv$ newtons where M kg is the mass of the parcel plus parachute and v ms^{-1} is the velocity after t seconds ($t \geq 10$ seconds).

Take the position of the helicopter to be the origin, the downwards direction as positive and the value of g, the acceleration due to gravity, as $10\ \text{ms}^{-2}$.

i. Show that the velocity of the parcel at the end of 10 seconds is $100\ \text{ms}^{-1}$ and the distance fallen at the end of 10 seconds is 500 metres. **2**

ii. Show that the velocity of the parcel after the parachute opens is given by **2**

$$v = 5 + 95e^{-2(t-10)}$$

for $t \geq 10$.

iii. Find x, the distance fallen as a function of t. **2**

(b) Two sequences $u_1, u_2, u_3, \cdots$ and $v_1, v_2, v_3, \cdots$ are given by

$$u_1 = 1, \;\; v_1 = 1 \;\text{ and}$$

$$u_{n+1} = u_n + 3v_n\,, \;\; v_{n+1} = 2u_n + 7v_n$$

for positive integers n.

i. Using Mathematical induction, prove that $2u_n^2 - 3v_n^2 + 6u_nv_n = 5$ for all positive integer n. **2**

The sequence $r_1\, r_2, r_3, \cdots$ is such that $r_n = \frac{u_n}{v_n}$ for positive integers n.

It is given that as $n \to \infty$, $v_n \to \infty$ and $r_n \to L$ for some real constant L.

ii. Using the result in (i) or otherwise, show that $L = \frac{1}{2}(\sqrt{15} - 3)$. **2**

Examination continues overleaf. . .

(c) Let $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ and $S_n = \sum_{r=1}^{n} \omega^r$, where n is a positive integer. **Marks**

It is given that $1 + \omega + \omega^2 = 0$. It is also given that $S_n = 0$ if n is multiple of 3. (DO NOT prove this).

i. Find S_n if n is a not multiple of 3. **2**

ii. Prove that there exists **no** integer m such that $(S_{2022} + S_{2023} + S_{2024})^m = 2$. **2**

iii. Find all positive integers k such that $(S_n)^k + (S_{n+1})^k + (S_{n+2})^k = 2$. **3**

End of paper

Sample solutions - ext 2 trial - 2023

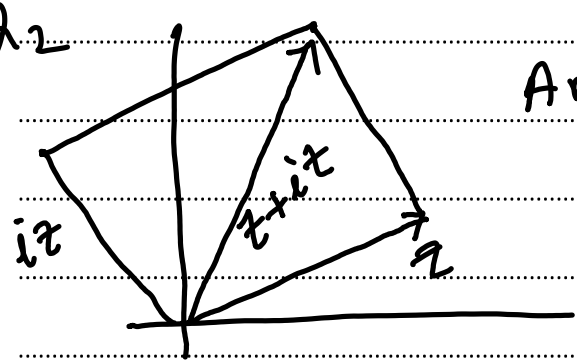
Q1 $4+3i = 5\left(\frac{4}{5} - \frac{3}{5}i\right)$

$= 5e^{i\theta}$, $\tan\theta = \frac{3}{4}$

$\doteqdot 5e^{0.644i}$

D

Q2



Area $= \frac{1}{2}|z||iz|$

$= \frac{1}{2}|z|^2$

C

Q3 D

Q4 $(x-1)^2 + (y-2)^2 + (z+2)^2 = 9$

$x^2 - 2x + y^2 - 4y + z^2 + 4z = 0$

D

Q5 $z^3 = -27i = 27e^{\frac{3\pi}{2}i}$

$z = 3e^{\frac{1}{3}\left(2k\pi + \frac{3\pi}{2}\right)i}$; $k = -2, -1, 0$

B

Q.6

$$g(x) = f(x) - f(-x)$$
$$g(-x) = f(-x) - f(x)$$
$$= -g(x)$$

C

Q7 $\int x\sqrt{1-x} = -\int (1-x-1)(1-x)^{1/2}\,dx$

$$= -\int (1-x)^{3/2} - (1-x)^{1/2}\,dx$$
$$= \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + c$$

D

Q8

$$c_4 z^4 + i c_3 z^3 + c_2 z^2 + i c_1 z + c_0 = 0$$
$$c_4 (iz)^4 - c_3 (iz)^3 - c_2 (iz)^2 + c_1 (iz) + c_0 = 0$$

Let $w = iz$

$$c_4 w^4 - c_3 w^3 - c_2 w^2 + c_1 w + c_0 = 0$$

$w = i(p+iq)$ is a solution

$= -q + ip$

$iz_1 = w_1 = -q - ip$ is also a sol^n

$\times(-i)$ $z_1 = -p + qi$ B

Q9 $r_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\-1\end{pmatrix}$

$r_2 = \begin{pmatrix}3\\2\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\0\\2\end{pmatrix}$

$1 = 3 + 2\mu \Rightarrow \mu = -1$

$\lambda = 2$

$-\lambda = -1 + 2\mu \qquad -2 \neq -1 - 2 \quad \therefore$ D

Q10

$$\int_{7\sqrt{6}}^{0} \frac{1}{9.8 + 0.1v^2}\, dv = \int_0^T -dt$$

$$-T = \frac{1}{\sqrt{9.8}\sqrt{0.1}}\left[\tan^{-1}\left(\frac{\sqrt{0.1}\,v}{\sqrt{9.8}}\right)\right]_{7\sqrt{6}}^{0}$$

$$T = \frac{10}{\sqrt{98}} \tan^{-1}\left(\frac{\sqrt{0.1} \times 7\sqrt{6}}{\sqrt{9.8}}\right)$$

$$= \frac{10}{7\sqrt{2}} \tan^{-1}(\sqrt{3}) = \frac{10\pi}{21\sqrt{2}} \text{ sec}$$

A

1 –	Ⓐ	Ⓑ	Ⓒ	●	6 –	Ⓐ	Ⓑ	●	Ⓓ
2 –	Ⓐ	Ⓑ	●	Ⓓ	7 –	Ⓐ	Ⓑ	Ⓒ	●
3 –	Ⓐ	Ⓑ	Ⓒ	●	8 –	Ⓐ	●	Ⓒ	Ⓓ
4 –	Ⓐ	Ⓑ	Ⓒ	●	9 –	Ⓐ	Ⓑ	Ⓒ	●
5 –	Ⓐ	●	Ⓒ	Ⓓ	10 –	●	Ⓑ	Ⓒ	Ⓓ

Q 11

a) $a+ib = \frac{2+4i}{1-i}(b+i) \times \frac{1+i}{1+i}$

$= \frac{2(1+2i)(1+i)}{2}(b+i)$

$= (-1+3i)(b+i)$

$= -b-3+(3b-1)i$

$a=-b-3 \quad a_n1 \qquad b= 3b-1$

$a=-\frac{7}{2} \qquad\qquad b=1/2$

b)

$\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2 \geq 0$

$\frac{a}{b}-2+\frac{b}{a} \geq 0$

$\frac{a}{b}+\frac{b}{a} \geq 2$

c) i) $z = -p+pi = p(-1+i)$

$= \sqrt{2}p\, e^{i\left(\frac{3\pi}{4}\right)}$

$w = \frac{\sqrt{2}\,\overline{z}}{z^4} = \frac{2p\, e^{-i\left(\frac{3\pi}{4}\right)}}{4p^4 e^{i(3\pi)}}$

$$= \frac{1}{2p^3} e^{-i\left(3\pi + \frac{3\pi}{4}\right)}$$

$$= \frac{\sqrt{2}}{p^3} e^{i\frac{\pi}{4}}$$

ii)

$$w^n = \left(\frac{\sqrt{2}}{p^3}\right)^n e^{i\left(\frac{n\pi}{4}\right)}$$

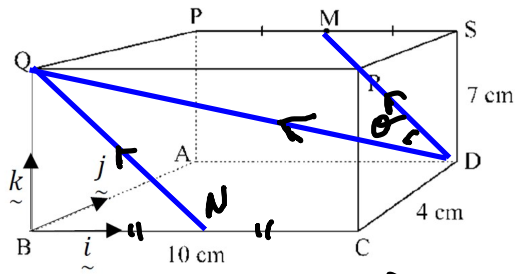
$$Re(w^n) = \left(\frac{\sqrt{2}}{p^3}\right)^n \cos\frac{n\pi}{4} = 0$$

$$\frac{n\pi}{4} = 2k\pi \pm \frac{\pi}{2}$$

$$n = 8k \pm 2$$

$k = 0$ $\quad n = 2$ is the smallest integer

d)



i)

$$\overrightarrow{DM} = \overrightarrow{NQ} = -5\underset{\sim}{i} + 7\underset{\sim}{k}$$

ii) $\overrightarrow{DQ} = \overrightarrow{DC} + \overrightarrow{CB} + \overrightarrow{BQ} = -10\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k}$

$$\cos\theta = \frac{\overrightarrow{DM} \cdot \overrightarrow{DQ}}{|\overrightarrow{DM}||\overrightarrow{DQ}|} = \frac{50 + 49}{\sqrt{74} \times \sqrt{165}}$$

$$\theta = 26.37098991 \doteqdot 26^\circ$$

e) $$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, \cos^3 x \, dx = \int_0^{\pi/2} \sqrt{\sin x}\,(1-\sin^2 x)\, d(\sin x)$$

$$= \left[\frac{2}{3}(\sin x)^{3/2} - \frac{2}{7}(\sin x)^{7/2}\right]_0^{\pi/2}$$

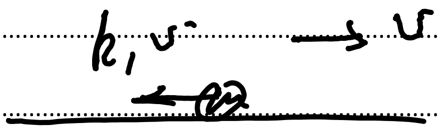
$$= \frac{2}{3} - \frac{2}{7} = \frac{8}{21}$$

Q2

a) i) $\forall x \in \mathbb{R} \quad (x^2 \leq 5) \Longrightarrow (x < 3)$

ii) $\exists x \in \mathbb{R}$ such that $(x \geq 3) \wedge (x^2 \leq 5)$

b)



i)

$F = ma \longrightarrow m\ddot{x} = -k_1 v$

$$\ddot{x} = -\frac{k_1}{m} v$$

$$\ddot{x} = -kv \quad \text{where } k = \frac{k_1}{m}$$

ii)

$$v \frac{dv}{dx} = -kv$$

$$\int_{20}^{10} 1 \, dv = \int_{0}^{80} -k \, dx$$

$$10 - 20 = -k(80 - 0)$$

$$-10 = -80k$$

$$k = \frac{1}{8}$$

iii)

$$\frac{dV}{dt} = -\frac{1}{8}V$$

$$\int_{20}^{10} \frac{1}{V}\,dV = \int_{0}^{T} -\frac{1}{8}\,dt$$

$$\ln 10 - \ln 20 = -\frac{T}{8}$$

$$T = 8\ln 2 \text{ sec}$$

c)

$$\frac{5(1-x)}{(1+x)(3-2x)} = \frac{5(x-1)}{(x+1)(2x-3)} = \frac{A}{x+1} + \frac{B}{2x-3}$$

$$5(x-1) = A(2x-3) + B(x+1)$$

$x=-1$ $\quad -10 = -5A \Rightarrow A = 2$

$x=\frac{3}{2}$ $\quad \frac{5}{2} = B\left(\frac{5}{2}\right) \Rightarrow B = 1$

$$\therefore \frac{5(1-x)}{(1+x)(3-2x)} = \frac{2}{x+1} + \frac{1}{2x-3}$$

OR

$$\frac{5(x-1)}{(x+1)(2x-3)} = \frac{5(x+1-2)}{(x+1)(2x-3)} = \frac{5}{2x-3} - \frac{10}{(x+1)(2x-3)}$$

$$= \frac{5}{2x-3} + 2\left[\frac{1}{x+1} - \frac{2}{2x-3}\right]$$

$$= \frac{1}{2x-3} + \frac{2}{x+1}$$

$$\int_0^1 \frac{5(1-x)}{(1+x)(3-2x)}\,dx = \int_0^1 \frac{1}{2x-3} + \frac{2}{x+1}\,dx$$

$$= \left[\frac{1}{2}\ln|2x-3| + 2\ln|x+1|\right]_0^1$$

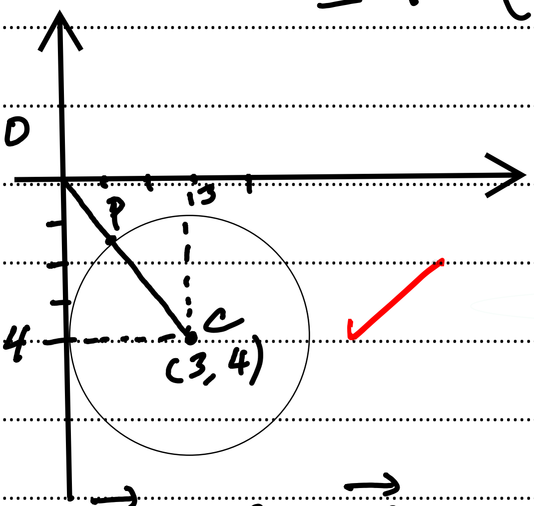
$$= 0 + 2\ln 2 - \left(\frac{1}{2}\ln 3\right)$$

$$= \ln 4 - \ln\sqrt{3}$$

$$= \ln\left(\frac{4}{\sqrt{3}}\right)$$

d)

i)



ii) $\vec{OP} = \frac{2}{5}\vec{OC}$

∴ the complex number represented by P

$$= \frac{2}{5}(3-4i)$$

$$= \frac{6}{5} - \frac{8}{5}i$$

Q.13

a) If both a and b odd or both even then $a+b$ is even

$\therefore$ one of them is odd and other is even

w.l.g take a is odd and b is even

then $\exists\, m, n \in \mathbb{Z}$ such that

$$a = 2m+1 \text{ and } b = 2n$$

$$a^2+b^2 = (2m+1)^2 + 4n^2$$
$$= 2(2m^2+2n^2+2m)+1$$
$$= 2k+1 \quad \text{where } k = 2m^2+2n^2+2m \in \mathbb{Z}$$

$\therefore a^2+b^2$ is odd

b)

i) $f(x) = x - \ln(1+x)$

$$f'(x) = 1 - \frac{1}{1+x} = \frac{x}{x+1} > 0 \text{ for } x \in (0,1)$$

and $f(0) = f'(0) = 0$

$$\therefore \quad f(0) < f(x) \quad \forall x \in (0,1)$$
$$0 < x - \ln(1+x)$$
$$\therefore \quad \ln(1+x) < x \quad \forall x \in (0,1) \quad \text{—} (1)$$

ii)

$$g(x) = x + \ln(1-x)$$

$$g'(x) = 1 - \frac{1}{1-x} = \frac{-x}{1-x}$$

$$= \frac{x}{x-1} < 0 \quad \forall x \in (0,1)$$

and $g(0) = 0$

$\therefore\ g(0) > g(x) \quad \forall x \in (0,1)$

$0 > x + \ln(1-x)$

$\therefore\ -\ln(1-x) > x \quad \forall x \in (0,1)$ —— ②

iii)

From ① & ②

$$\ln(1+x) < x < -\ln(1-x)$$

sub $x = \frac{1}{n}$ for $n > 1$

$$\ln\left(1+\frac{1}{n}\right) < \frac{1}{n} < -\ln\left(1-\frac{1}{n}\right)$$

$$\ln(1+n) - \ln n < \frac{1}{n} < \ln n - \ln(n-1)$$

iv)

$n=2 \quad \ln 3 - \ln 2 < \frac{1}{2} < \ln 2 - \ln 1$

$n=3 \quad \ln 4 - \ln 3 < \frac{1}{3} < \ln 3 - \ln 2$

$\vdots$

$n=1000 \quad \ln 1001 - \ln 1000 < \frac{1}{1000} < \ln 1000 - \ln 999$

$+$

$$\ln 1001 - \ln 2 < \sum_{k=2}^{1000} \frac{1}{k} < \ln 1000$$

$$6.21560\ldots = \ln \frac{1001}{2} < \sum_{k=2}^{1000} < \ln 1000 = 6.9077\ldots$$

$$6.21 < \sum_{k=2}^{1000} < 6.91$$

c)

$$3z^3 + 13z^2 + 20z + 14 = 0$$

roots : $-1+i, -1-i, \alpha$

$$-2 + \alpha = -\frac{13}{3}$$

$$\alpha = -\frac{7}{3}$$

Q14

a) i) $x = -y = \frac{z}{2} = \lambda$

$l_1 : \underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda \\ -\lambda \\ 2\lambda \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ ✓

ii)

$l_2 : \underset{\sim}{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ✓

iii)

$\cos\theta = \dfrac{1+0+2}{\sqrt{6}\,\sqrt{2}} = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2}$

$\theta = 30^\circ$ ✓

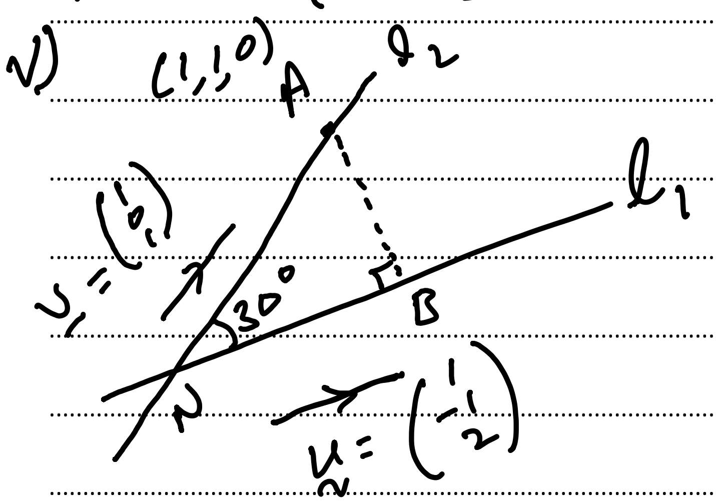
iv)

$\lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+\mu \\ 1 \\ \mu \end{pmatrix} \Rightarrow \begin{array}{l} \lambda = 1+\mu \\ \lambda = -1 \\ \mu = -2 \end{array}$

<u>check</u> $2\lambda = \mu$ when $\lambda = -1$, $\mu = -2$

$\therefore N\,(-1, 1, -2)$ ✓

v)



<u>method 1</u>

$AN = \sqrt{2^2+0^2+2^2}$
$= 2\sqrt{2}$ ✓

$AB = 2\sqrt{2}\sin 30^\circ$
$= \sqrt{2}$ units ✓

<u>method 2</u> $\overrightarrow{NA} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

$$\overrightarrow{NB} = \text{Proj}_{\underset{\sim}{u}} \overrightarrow{NA}$$

$$= \frac{\overrightarrow{NA} \cdot \underset{\sim}{u}}{|\underset{\sim}{u}|^2} \underset{\sim}{u}$$

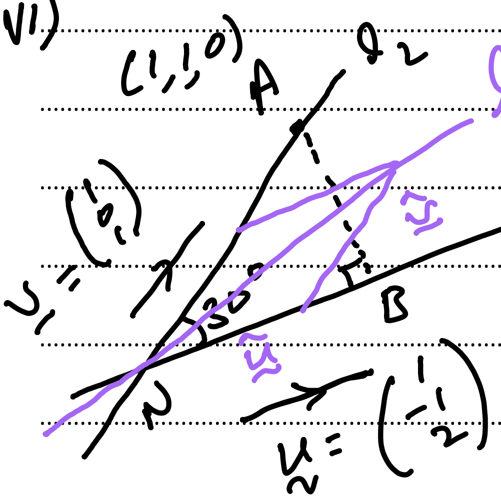
$$= \frac{2+6}{8} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \checkmark$$

$$\overrightarrow{AB} = \overrightarrow{AN} + \overrightarrow{NB}$$

$$= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{2} \checkmark$$

vi)



Direction vector of l_3

$$= \hat{\underset{\sim}{u}} + \hat{\underset{\sim}{v}}$$

$$= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \checkmark$$

$$= \frac{1}{\sqrt{6}} \begin{pmatrix} 1+\sqrt{3} \\ -1 \\ 2+\sqrt{3} \end{pmatrix}$$

Equation of l_3

$$\underset{\sim}{r} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1+\sqrt{3} \\ -1 \\ 2+\sqrt{3} \end{pmatrix} ; \lambda \in \mathbb{R} \checkmark$$

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \sin x \sin^{n-1} x \, dx$$

$$= \left[\sin^{n-1} x \, (-\cos x)\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2} x \cos^2 x \, dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, (1 - \sin^2 x) \, dx$$

$$I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$I_n (1 + n - 1) = (n-1) I_{n-2}$$

$$n I = (n-1) I_{n-2}$$

$$x_n = n I_n I_{n-1}$$

$$= (n-1) I_{n-2} I_{n-1}$$

$$= (n-1) I_{n-1} I_{n-2} = x_{n-1}$$

$$x_1 = 1 I_1 I_0$$

$$I_0 = \int_0^{\frac{\pi}{2}} 1 \, dx = \frac{\pi}{2}$$

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx = \left[-\cos x\right]_0^{\frac{\pi}{2}} = 1$$

$$x_1 = \frac{\pi}{2}$$

but by ii $x_n = x_{n-1} = x_{n-2} = \cdots = x_2 = x_1$
$= \frac{\pi}{2}$

$0 \le \sin x \le 1 \quad \forall x \in [0, \frac{\pi}{2}]$
$\therefore \sin^n x \le \sin^{n-1} x \quad \forall x \in [0, \frac{\pi}{2}]$

$\therefore I_n \le I_{n-1}$

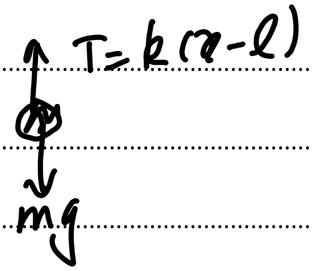
$\frac{\pi}{2} = x_n = n I_n I_{n-1}$ by ii
$\therefore \frac{\pi}{2} \geq n I_n I_n$ by iii
$I_n^2 \le \frac{\pi}{2n}$
$I_n \le \sqrt{\frac{\pi}{2n}}$ — (*)

$\frac{\pi}{2} = x_{n+1} = (n+1) I_{n+1} I_n$
$\le (n+1) I_n I_n$
$I_n^2 \geq \frac{\pi}{2(n+1)}$
$I_n \geq \sqrt{\frac{\pi}{2(n+1)}}$ — (* *)

From $\cancel{(**)}$, (*,*)
$\sqrt{\frac{\pi}{2(n+1)}} \le I_n \le \sqrt{\frac{\pi}{2n}}$

Q15

a)



i) $F = ma \downarrow$

$$m\ddot{x} = mg - k(x-l)$$

$$\ddot{x} = g - \frac{k}{m}(x-l)$$

$$= -\frac{k}{m}\left(x - l - \frac{mg}{k}\right)$$

$$= -\frac{k}{m}\left(x - \left(\frac{mg}{k} + l\right)\right)$$

ii)

$$x = a\cos\left(\sqrt{\frac{k}{m}}\,t + \alpha\right) + \frac{mg}{k} + l$$

$$\dot{x} = -a\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}\,t + \alpha\right)$$

$$\ddot{x} = -\frac{ak}{m}\cos\left(\sqrt{\frac{k}{m}}\,t + \alpha\right)$$

$$= -\frac{k}{m}\left(x - \left(\frac{mg}{k} + l\right)\right)$$

$\therefore\ x = a\cos\left(\sqrt{\frac{k}{m}}\,t + \alpha\right) + \frac{mg}{k} + l$ satisfies the DE.

iii)

$$\frac{mg}{k} + l = \frac{4 \times 9.8}{49} + 5 = 5.8$$

$$\ddot{x} = -\frac{49}{4}(x - 5.8)$$

$\therefore$ Centre of motion at $x = 5.8$ m.

iv)

$$\text{period} = P = \frac{2\pi}{\sqrt{\frac{49}{4}}} = \frac{4\pi}{7} \text{ sec}$$

method 1

$t = 0 \quad x = 6\text{m}, \quad \dot{x} = 3.5 \text{ ms}^{-1}$

$$6 = a\cos\alpha + 5.8$$

$$0.2 = a\cos\alpha \quad — \text{①}$$

$$3.5 = -a\sqrt{\frac{49}{4}}\sin(\alpha)$$

$$1 = -a\sin\alpha \quad — \text{②}$$

$①^2 + ②^2 \qquad 0.2^2 + 1^2 = a^2$

$$a = \sqrt{1.04} = 1.01980\ldots$$

$$\doteqdot 1.02 \text{ m}$$

method 2

$$v^2 = n^2(a^2 - (x - c)^2)$$

$$v^2 = \frac{49}{7}(a^2 - (x - 5.8)^2)$$

sub $x = 6 \qquad v = 3.5$

$$3.5^2 = \frac{49}{7}\left(a^2 - (6-5.8)^2\right)$$

$$1 = a^2 - (0.2)^2 \Rightarrow a^2 = 1.04$$

$$a = \sqrt{1.04}$$

Method 3

$$\ddot{x} = -\frac{49}{4}(x - 5.8)$$

$$v\frac{dv}{dx} = -\frac{49}{4}(x-5.8)$$

$$\int_{3.5}^{0} v\,dv = -\frac{49}{4}\int_{6}^{x_1}(x-5.8)\,dx$$

$$\left[\frac{v^2}{2}\right]_{3.5}^{0} = -\frac{49}{4}\left[\frac{x^2}{2} - 5.8x\right]_6^{x_1}$$

$$-\frac{3.5^2}{2} = -3.5^2\left[\frac{x_1^2}{2} - 5.8x_1 - \left(\frac{36}{2} - 5.8\times 6\right)\right]$$

$$\frac{1}{2} = \frac{x_1^2}{2} - 5.8x_1 + 16.8 \Rightarrow x_1^2 - 11.6x_1 + 32.6 = 0$$

$$x_1 = 6.81982,\ 4.7802$$

$$\therefore a = 6.81982 - 5.8 = 1.019\ldots \doteqdot 1.02$$

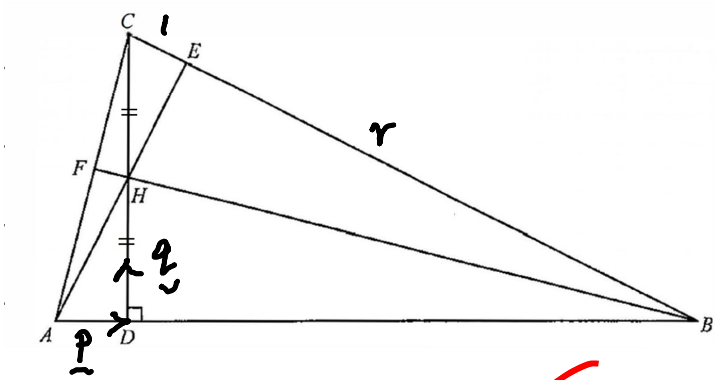
v)

Extrema are at $x = 5.8 \pm \sqrt{1.04}$ m

At $x = 5.8 + \sqrt{1.04}$ $\quad T = 49(5.8 + \sqrt{1.04} - 5)$

$$= 89.17 \text{ N}$$

At $x = 5.8 - \sqrt{1.04}$ $\quad T = -10.77$ N b)



$\overrightarrow{AB} = \lambda \underset{\sim}{p}$

$\overrightarrow{BE} = r\,\overrightarrow{EC}$

i) $\overrightarrow{AC} = \underset{\sim}{p} + 2\underset{\sim}{q}$

ii) $\overrightarrow{AE} = \overrightarrow{AC} + \overrightarrow{CE}$

$= \overrightarrow{AC} + \frac{1}{1+r}\overrightarrow{CB}$

$= \overrightarrow{AC} + \frac{1}{1+r}\left(\overrightarrow{CA} + \overrightarrow{AB}\right)$

$= \left(1 - \frac{1}{1+r}\right)\overrightarrow{AC} + \frac{1}{1+r}\overrightarrow{AB}$

$= \frac{r}{r+1}\left(\underset{\sim}{p} + 2\underset{\sim}{q}\right) + \frac{1}{1+r}\left(\lambda \underset{\sim}{p}\right)$

$= \dfrac{(r+\lambda)\underset{\sim}{p} + 2r\underset{\sim}{q}}{1+r}$

iii)

Since A, H and E Collinear

$\overrightarrow{AH} = \mu \overrightarrow{AE}$

$\underset{\sim}{p} + \underset{\sim}{q} = \frac{\mu}{1+r}\left[(r+\lambda)\underset{\sim}{p} + 2r\underset{\sim}{q}\right]$

$\therefore \quad 1 = \frac{\mu(r+\lambda)}{1+r}$ and $1 = \frac{2r\mu}{1+r}$

$$\therefore\ \mu(r+\lambda) = 2r\mu$$
$$\lambda + r = 2r$$
$$\lambda = r$$

iv)

$$\overrightarrow{AH} \cdot \overrightarrow{BC} = 0$$
$$\overrightarrow{AH} \cdot (\overrightarrow{BA} + \overrightarrow{AC}) = 0$$
$$(\underset{\sim}{p}+\underset{\sim}{q}) \cdot (-\lambda \underset{\sim}{p} + \underset{\sim}{p} + 2\underset{\sim}{q}) = 0$$
$$(\underset{\sim}{p}+\underset{\sim}{q}) \cdot ((1-\lambda)\underset{\sim}{p} + 2\underset{\sim}{q}) = 0$$
$$(1-\lambda)|\underset{\sim}{p}|^2 + (2+1-\lambda)\,\underset{\sim}{p}\cdot\underset{\sim}{q} + 2|\underset{\sim}{q}|^2 = 0$$
$$1-\lambda + (3-\lambda)\,0 + 2(4) = 0$$
$$\lambda = 9$$

v)

Let $\overrightarrow{AF} = m\overrightarrow{AC}$ and $\overrightarrow{BH} = \gamma\overrightarrow{BF}$

$$\overrightarrow{AF} = m(\underset{\sim}{p} + 2\underset{\sim}{q})$$

$$\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BF}$$
$$= \overrightarrow{AB} + \frac{1}{\gamma}\overrightarrow{BH}$$
$$= \overrightarrow{AB} + \frac{1}{\gamma}(\overrightarrow{BD} + \overrightarrow{DH})$$
$$= 9\underset{\sim}{p} + \frac{1}{\gamma}(-8\underset{\sim}{p} + \underset{\sim}{q})$$
$$= \left(9 - \frac{8}{\gamma}\right)\underset{\sim}{p} + \frac{1}{\gamma}\underset{\sim}{q}$$

$\therefore\ m\underset{\sim}{p} + 2m\underset{\sim}{q} = \left(9 - \frac{8}{r}\right)\underset{\sim}{p} + \frac{1}{r}\underset{\sim}{q}$

$m = 9 - \frac{8}{r}$ and $2m = \frac{1}{r}$

$m = 9 - 8(2m)$

$17m = 9$

$m = \frac{9}{17}$

$\overrightarrow{AF} = \frac{9}{17}\overrightarrow{AC}$ $\therefore\ AF : FC = 9 : 8$

Method 2 Let $\overrightarrow{AF} = m\overrightarrow{AC} = m(\underset{\sim}{p} + 2\underset{\sim}{q})$

$\overrightarrow{AC} \cdot \overrightarrow{FH} = 0 \Rightarrow \overrightarrow{AC} \cdot (\overrightarrow{AH} - \overrightarrow{AF}) = 0$

$(\underset{\sim}{p} + 2\underset{\sim}{q}) \cdot (\underset{\sim}{p} + \underset{\sim}{q} - m(\underset{\sim}{p} + 2\underset{\sim}{q})) = 0$

$(\underset{\sim}{p} + 2\underset{\sim}{q})((1-m)\underset{\sim}{p} + (1-2m)\underset{\sim}{q}) = 0$

$(1-m)|\underset{\sim}{p}|^2 + (1-2m)\underset{\sim}{p} \cdot \underset{\sim}{q} + 2(1-m)\underset{\sim}{p} \cdot \underset{\sim}{q} + 2(1-2m)|\underset{\sim}{q}|^2 = 0$

$(1-m)1 + 2(1-2m) \cdot 4 = 0$

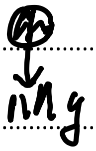
$1 - m + 8 - 16m = 0 \Rightarrow m = \frac{9}{17}$

$\therefore\ AF : FC = 9 : 8$

Q16

a)

i)



$F = ma$ $\downarrow$

$m\ddot{x} = mg$

$\ddot{x} = \dfrac{dv}{dt} = g = 10$

$$\int_0^v 1\,dv = \int_0^t 10\,dt$$

$v = 10t$

$t = 10$ $\quad v = 100 \text{ ms}^{-1}$

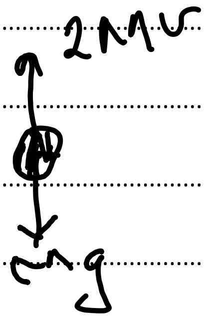
$\dfrac{dx}{dt} = 10t$

$$\int_0^x dx = \int_0^{10} 10t\,dt$$

$x = [5t^2]_0^{10} = 500 \text{ m}$

ii)

For $t \geq 10$



$F = ma$ $\downarrow$

$m\ddot{x} = mg - 2mv$

$\ddot{x} = g - 2v$

$= 10 - 2v$

$= -2(v - 5)$

$5 \leq v \leq 100$

$$\int_{100}^{v} \frac{1}{v-5}\,dv = \int_{10}^{t} -2\,dt$$

$$\Big[\ln(v-5)\Big]_{100}^{v} = -2(t-10)$$

$$\ln(v-5) - \ln(95) = -2(t-10)$$

since $v-5>0$

$$\ln\left(\frac{v-5}{95}\right) = -2(t-10)$$

$$v = 5 + 95e^{-2(t-10)}$$

iv)

$$\frac{dx}{dt} = 5 + 95e^{-2(t-10)}$$

$$\int_{500}^{x} dx = \left[5t - \frac{95}{2}e^{-2(t-10)}\right]_{10}^{t}$$

$$x - 500 \quad 5t - \frac{95}{2}e^{-2(t-10)} - \left(50 - \frac{95}{2}\right)$$

$$x = 5t - \frac{95}{2}e^{-2(t-10)} + \frac{995}{2}$$

$S(n): 2U_n^2 - 3V_n^2 + 6U_nV_n = 5$

prove for $S(1)$

$$L.H.S = 2U_1^2 - 3V_1^2 + 6U_1V_1$$
$$= 2 - 3 + 6$$
$$= 5 = R.H.S$$

$\therefore S(1)$ is true

Assume $S(k)$ is true

i.e. $2U_k^2 - 3V_k^2 + 6U_kV_k = 5$

prove for $S(k+1)$

$$L.H.S = 2U_{k+1}^2 - 3V_{k+1}^2 + 6U_{k+1}V_{k+1}$$
$$= 2(U_k+3V_k)^2 - 3(2U_k+7V_k) + 6(U_k+3V_k)(2U_k+7V_k)$$
$$= 2U_k^2 + 12U_kV_k + 18V_k^2 - 12U_k^2 - 84U_kV_k - 147V_k^2 + 12U_k^2 + 78U_kV_k + 126V_k^2$$
$$= 2U_k^2 + 6U_kV_k - 3V_k^2$$
$$= 5 \text{ by assumption.}$$
$$= R.H.S$$

$\therefore S(k+1)$ is true

$\therefore$ by MI, $S(n)$ is true for all $n \in \mathbb{N}$

$$r_n = \frac{u_n}{v_n}, \quad \lim_{n\to\infty} v_n = 0 \text{ and } \lim_{n\to\infty} r_n = L$$

$\div v_n^2$ $\quad 2u_n^2 - 3v_n^2 + 6u_n v_n = 5$

$$2r_n^2 - 3 + 6r_n = \frac{5}{v_n}$$

Take limit as $n \to \infty$ both sides

$$2L^2 - 3 + 6L = \lim_{n\to\infty} \frac{5}{v_n} = 0$$

$$2L^2 + 6L - 3 = 0$$

$$L = \frac{-6 \pm \sqrt{36 + 4\times 2\times 3}}{4}$$

$$= \frac{-3 \pm \sqrt{15}}{2}$$

Since $u_n, v_n > 0$, $r_n > 0 \ \forall n$ $\therefore L > 0$

$$\therefore L = \frac{1}{2}(\sqrt{15} - 3)$$

c)

i) $\omega = \text{cis}\,\frac{2\pi}{3}$; $S_n = \sum_{r=1}^{n} \omega^k$

$S_n = \omega + \omega^2 + \dots + \omega^n$ and $\omega^3 = 1$

$= \frac{\omega(\omega^n - 1)}{\omega - 1}$

if n is not multiple of 3 then

$n = 3m+1$ OR $n = 3m+2$ for some $m \in \mathbb{Z}$

If $n = 3m+1$

$S_n = \frac{\omega(\omega^{3m} \times \omega^1 - 1)}{\omega - 1} = \frac{\omega(\omega - 1)}{\omega - 1}$

$= \omega$

If $n = 3m+2$ $\quad S_n = \frac{\omega(\omega^{3m}\omega^2 - 1)}{\omega - 1}$

$= \omega(\omega + 1)$

$= \omega + \omega^2$

$= -1$

Assume there exist an integer m

such that $(S_{2022} + S_{2023} + S_{2024})^m = 2$

$2022 = 3 \times 674 \Rightarrow S_{2022} = 0$

$2023 = 3 \times 674 + 1 \Rightarrow S_{2023} = \omega$

$2024 = 3 \times 674 + 2 \Rightarrow S_{2024} = -1$

$\therefore (0 - 1 + \omega)^m = 2$

$$\left(-1-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^m = 2$$

$$\left(-\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)^m = 2$$

$$\left|-\frac{3}{2}+\frac{\sqrt{3}}{2}i\right| = \sqrt{\frac{9}{4}+\frac{3}{4}} = \sqrt{3}$$

$$\sqrt{3}^{\,m} = 2$$

$$3^m = 2^2$$

$\#$ since L.H.S is multiple of 3 and R.H.S $= 4$

$\therefore \nexists\ m \in \mathbb{Z}$ such $(S_{2022}+S_{2023}+S_{2024})^m = 2$

$$(S_n)^k + (S_{n+1})^k + (S_{n+2})^k = 2$$

$$\{S_n, S_{n+1}, S_{n+2}\} = \{0, -1, w\}$$

$$\therefore\ 0^k + (-1)^k + w^k = 2$$

If k is odd $\quad w^k = 3$

$$\Rightarrow |w|^k = 3 \Rightarrow 1 = 3\ \#$$

If k is even $\quad w^k = 1$

$$\operatorname{Cis}\frac{2k\pi}{3} = 1 = \operatorname{Cis} 0$$

$$\frac{2k\pi}{3} = 2m\pi \quad \text{for } m = 0, \pm 1, \pm 2, \ldots$$

$$k = 3m$$

$\therefore$ k is even and multiple of 3

$\therefore\ k = 6n \quad$ for $n = 0, \pm 1, \pm 2, \ldots$